Probabilistic description of chaotic deterministic systems

Research internship — Academic year 2023/2024

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The dynamics of chaotic systems exhibit an extreme sensitivity to initial conditions. Even the slightest variations between initially close trajectories lead to exponential separation over time, rendering long-term predictions exceedingly challenging. Famous examples include Lorenz's system modeling atmospheric convection [1] and the long-term motion of the inner planets in the Solar System [2]. Despite the deterministic nature of these systems, their behavior ultimately evolves into a state of actual randomness on long time scales. Consequently, it becomes necessary to establish a statistical description, framed in terms of a probability density defined over the phase space of the dynamics.

In the conventional Monte Carlo approach, the probability density of physical observables is estimated from a large ensemble of numerical integrations of the equations of motion. The internship opportunity presented here aims to explore an alternative approach to address the probabilistic description of chaotic dynamics.

Liouville's theorem is a fundamental result of Hamiltonian mechanics, establishing the time invariance of the phase-space probability density along the system trajectories. This result is expressed via the Liouville equation $\partial\rho/\partial t + \{H, \rho\} = 0$, where ρ is the probability density, H represents the Hamiltonian and $\{\cdot, \cdot\}$ stands for the Poisson bracket. This equation forms the foundation of statistical mechanics, and its formal solution is $\rho(t) = \mathcal{L}^t \rho(0)$, where $\mathcal{L}\cdot = \exp(-\{H, \cdot\})$. In this probabilistic framework, the evolution of a nonlinear dynamical system can be reformulated in terms of eigenvalues and eigenvectors of the linear operator $\mathcal{L}$, as in the case of a quantum system governed by Schrödinger equation.

In 1960, S. Ulam proposed a numerical implementation of this approach, which involves the partitioning of phase space into elementary cells to construct a finite-dimensional approximation of $\mathcal{L}$ [3]. However, the computational complexity of this method increases exponentially with the dimension of the phase space. Building upon a recent generalization of Ulam's method [4], the internship is dedicated to examining the feasibility of this approach for Hamiltonian systems with a moderate number of degrees of freedom, beginning with the well-known Henon-Heiles system [5].

Type of internship: Theory. Numerical computation.

Eligibility: Primarily open to graduate students in their first or second year of Master's studies.

Prerequisites: Knowledge of Hamiltonian mechanics and linear algebra. Good coding experience, with proficiency in programming languages such as Python and C. Familiarity with statistical physics and quantum mechanics would help.

Internship period: 4 months at least, from February-March to June-July 2024. The intern will benefit from a monthly compensation and partial reimbursement of public transportation costs.

References

[1] E. N. Lorenz, Journal of Atmospheric Sciences **20**, 130 (1963).
[2] J. Laskar, Nature **338**, 237 (1989).
[3] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience Tracts in Pure and Applied Mathematics (Interscience Publishers, 1960).
[4] K. M. Frahm and D. L. Shepelyansky, European Physical Journal B **76**, 57 (2010).
[5] M. Henon and C. Heiles, The Astronomical Journal **69**, 73 (1964).